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THE ROLE OF BONDS IN A MONETARY GROWTH MODEL

by

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THE UNIVERSITY OF ALBERTA

FACULTY OF GRANDUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled THE ROLE OF BONDS IN A MONETARY GROWTH MODEL submitted by EBRAHIM ASLANI-AMOLI in partial fulfilment of the requirements for the degree of Master of Arts.

TO

SEDDEGHEH and ISMAIL

ABSTRACT

The growth model initiated by R. M. Solow dealt only with the real sector of an economy. The neoclassical monetary growth model dealt with outside money and its effects on the real sector of an economy. However, outside money or government debt itself consists of two components: high-powered money and government bonds. The outstanding government debt is changed by a direct increase or decrease in high-powered money, and by an increase or decrease in the money supply generated by open market operations. The second component of outside money was not included in the earlier neoclassical monetary growth models. This study attempts to take into account both components of outside money. When the second component is included, it affects disposable income which, in turn, affects the saving and investment functions.

The question of stability is examined in the short-run and long-run dynamic models with three assets; money, bonds, and physical capital stock. It has been found that the stability of the dynamic system depends on the interest elasticities of demand for money and bonds, and on the natural rate of growth of an economy. This study suggests that the monetary authorities could select the ratio of money to bonds in such a way that the economy would move into the region of the stable growth path.

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LIST OF NOTATION

a_i = an expression used in the Routh Theorem,

a_{ij} = an element of non-singular matrix E,

a = the real value of all securities (whether held by the private sector or by the central bank),

b = B/rPL: real bond holdings per capita,

B = the number of nominal bonds outstanding, paying one dollar per coupon,

$\dot{B}$ = the rate of new issue of government bonds,

C = the rate of real consumption expenditure,

cY = the real value of business profits,

g = the rate of growth of the financial market,

G = the rate of real government expenditures on goods and services,

I = the rate of real investment expenditure,

k = K/L: capital-labour ratio,

k_1 = the speed of the price movement in the commodity and service market,

k_2 = the speed of the price movement in the securities market,

K = the level of real stock of physical capital,

L = labour force available at time t,

m = M/PL: real money holdings per capita,

M = the nominal stock of high-powered money,

$\dot{M}$ = the rate of new issue of money in nominal terms,

n = the natural rate of growth of the labour force,

P = the general price level,

r = the nominal market rate of interest,

r^o = the real market rate of interest,

s = the marginal and average propensity to save out of disposable income,

S = the level of real saving,

t = the time period,

U = the marginal and average tax rate,

V^o = the rate of nominal transfer payments excluding the interest payment on the outstanding government bonds,

V = the rate of real tax revenue,

w = the level of real wages,

W = the level of real wealth in the private sector,

y = the rate of real national output per capita,

Y = net national product in real terms,

yd = the rate of real disposable income per capita,

Yd = the rate of real disposable income,

β = the proportion of the total supply of securities held by the private sector as opposed to that held by the central bank,

γ = the "expectation coefficient" of changes in the price level,

μ = the rate of growth of nominal money,

ε = the speed of adjustment in the asset markets,

δ = the rate of growth of nominal bonds,

η_b = the elasticity of demand for bonds with respect to the rate of interest,

η_m = the elasticity of demand for money with respect to the rate of interest,

θ = $\frac{\dot{M}}{M} = \frac{\dot{B}}{B}$: the rate of money or bond expansion,

θ_1 = $\frac{m}{b}$: the ratio of government debt, i.e., the ratio of money to bonds,

ρ = the real rate of return on capital, the marginal product of capital,

π = the expected rate of change in the price level,

$*$ = over a symbol stands for the equilibrium value,

$\cdot$ = (a dot) over a symbol stands for time derivative.

CHAPTER I

INTRODUCTION

This thesis deals with the analysis of the monetary growth model and follows the general form of Tobin's work. Despite the large volume of literature on the monetary growth model, little has been done to explain the role of bonds and the interest rate in the growth process. In this study, money is outside money, and new issues of money and bonds are created to finance the government budget deficits. We assume in this study that the government expenditures on goods and services, G, is equal to the government revenue from taxation, V. The government budget deficits then consist of the transfer payments and the interest payments to the bond holders.

The study begins in Chapter II, where we review the basic neoclassical growth model and the role of outside money in the model. Since modern economies cannot ignore the role of the interest rate and its effects on the real sector of the economy, in the second section of Chapter II we examine the role of bonds in a growing economy. Unfortunately, there is not yet as much work done in this area.

In Chapter III, we introduce, therefore, government bonds into the neoclassical monetary growth model in a very simply way, and we specify a growth model with three assets, namely, private physical capital, money, and government bonds. The model also contains an aggregate production function, a growing labour force, and behavior equations for asset-holdings and private savings. Disposable income

has an important role in this model and consists of real income plus the increment in government debt.

In Chapter IV, the main part of this thesis, the dynamic path of the model will be examined. In particular, the existence of steady equilibrium growth paths and their stability conditions will be analysed for both the short-run and long-run dynamic systems. Since the method of stability employed here is rather mathematical in nature, occasional mathematical presentations have been unavoidable throughout the thesis. The Routh theorem has been used in order to find whether or not the dynamic system is stable.

The results of our study are presented in Chapter V, the closing chapter. Here we not only present the results, but also indicate that some of the assumptions could be relaxed in order to have better results in the sense to approximate reality more closely.

CHAPTER II

REVIEW OF MONEY AND BONDS IN GROWTH THEORY

The Role of Money and in a Growing Economy

The basic assumptions of the Solovian neoclassical growth model[1] for an economy that produces a single homogeneous output, Y, which can be either consumed or saved and invested, are as follows:

(1) The rate of growth of the labour force is given exogenously as n.

(2) The production function, $Y = F(K,L)$, is linear and homogeneous. It has the special properties of substitutability of labour and capital, and exhibits diminishing marginal returns to both factors.

(3) Investment is assumed to be always equal to saving, and saving is a fixed fraction of output.

Since the production function is linearly homogeneous, we can show output per capita as a function of capital per capita:

(2.1) $$y = f(k), \text{ where } y = Y/L, \text{ and } k = K/L .$$

Any growth model must be able to answer the following questions. Do long-run equilibrium values of capital per capita, k,

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[1] R. M. Solow, "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics 70 (February 1956): 65-94. See also R. M. Solow, Growth Theory: An Exposition (London: Oxford University Press, 1970), pp. 1-57.

and output per capita, y, exist? Are these values stable? What are the values of wages and profits? What is the distribution of output between wages and profits? What are the values of consumption and saving per capita?

Indeed, the neoclassical growth model tends toward a long-run equilibrium of steady state growth starting from any initial values different from those required for steady state growth paths. Given the values of capital and labour available for employment under the condition of profit maximization, i.e.,

$$\rho = \frac{dy}{dk} = f'(k) \tag{2.2}$$

$$w = \frac{dy}{dL} = f(k) - f'(k) \cdot k \tag{2.3}$$

$$y = f(k) = w + \rho k \ , \tag{2.4}$$

the combination of factors is assumed to be such that there will always be full employment of available capital and labour at an appropriate level of wages, w (in real terms), and rate of profit, ρ. The properties of the production function will guarantee the existence of a unique solution at w and ρ greater than zero with any arbitrary K and L. Since saving equals income minus consumption, and saving equals investment, as an equilibrium condition, we have no problem of unemployment due to ineffective demand, as we have in Keynesian theory. From $k = K/L$, by logarithmic differentiation with respect to time we get:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \ , \tag{2.5}$$

and since $\dot{K} = I = S = sY = sF(K,L)$, we have

$$\frac{\dot{K}}{K} = \frac{sF(K,L)}{K} = \frac{sf(k)}{k} .$$

Therefore,

$$\frac{\dot{k}}{k} = \frac{sf(k)}{k} - n . \tag{2.6}$$

Eq (2.6) is Solow's fundamental equation. Steady state full employment growth at a constant capital-labour ratio requires that $\dot{k} = 0$; thus $\frac{f(k)}{k} = \frac{n}{s}$. The "well-behaved" production function ensures that there always exists a unique value of k^* as a solution to Eq (2.6), the long-run equilibrium capital-labour ratio, at which equilibrium output per capita is y^*, and Y, K, and L all grow at the same rate n. It is easy to explain that, starting from any value at which $k_0 \neq k^*$, an economy will eventually converge to a steady state growth path. Beginning at any point, such as $k_o < k^*$, saving out of full employment income exceeds the investment required at the existing capital-labour ratio to provide employment for the increment in the labour force. In other words, the existing capital-labour ratio is too low. In Harrod's term we have a situation where the warranted growth rate exceeds the natural growth rate. Since it is assumed that $S = I$, the capital-labour ratio must rise, and, given a competitive market, the wage rate rises and the profit rate falls. Under the assumption of profit maximization (or cost minimization), the producers choose a technique with a higher capital-labour ratio. The value of w rises to the point where all the available stock of capital is fully utilized. This process continues during

the subsequent periods, and eventually the economy will reach that point where the capital-labour ratio, k^*, is such that the available saving is just enough to employ the increment in the labour force with the same capital intensity. At such a point the warranted growth rate will be equal to the natural growth rate and the solution will become consistent with a steady state. Alternatively, if the natural growth rate is greater than the warrented growth rate, a similar process will work in the opposite direction, i.e., w falls and ρ goes up.

Because of the properties of the production function, capital and labour can be substituted so long as w and ρ are assumed to be flexible. Therefore, the full employment of available capital and labour is always guaranteed regardless of the size of the labour force and the stock of capital. Moreover, such a substitution can go on until the steady state is reached.

These results follow from the assumption of a single output economy in which planned investment is equal to planned savings, i.e., markets are always in equilibrium. The only form in which wealth can be held is real capital. The model deals with only the real sector of an economy. Since modern economies are monetary economies, the introduction of money into the growth model is a step towards reality. Tobin introduces money into the neoclassical growth model.[1] The introduction of outside money into the model raises the

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[1] J. Tobin, "Dynamic Aggregative Model," <u>Journal of Political Economy</u> 63 (April 1955): 103-15. See also J. Tobin, "Money and Economic Growth," <u>Econometrica</u> 33 (October 1965): 671-84.

question of whether or not the money has any effect on the real variables of a growing economy, i.e., the question of whether money is neutral. In static analysis, if wealth is not an argument in the saving function, and if prices are flexible, the quantity theory suggests that doubling the stock of money will cause the price level to double with no effect on the real sector of the economy. In growth theory, the question of the neutrality of money is related, not to the absolute level of the money supply, but to its rate of change over time. That is, if money is neutral, changes in the rate of growth of the nominal money supply will have no effect on the equilibrium growth path of output and consumption per capita. From Solow's fundamental equation, Eq (2.6), we get:

$$\dot{k} = sf(k) - nk \tag{2.7}$$

$$\frac{I}{L} = nk + \dot{k} \tag{2.8}$$

where I is investment, and $\frac{I}{L} = sf(k)$. Output per capita can be written as

$$y = \frac{C}{L} + \frac{I}{L} \tag{2.9}$$

where C is consumption. By substituting (2.8) into (2.9) we get

$$\dot{k} = (y - nk) - \frac{C}{L} \ .$$

$(y - nk)$ is the per capita output available for consumption and for the change in capital-labour ratio, k. If money is not neutral, it

can affect either $\frac{C}{L}$ or $(y - nk)$.

However, the main results of Tobin's work are: (a) the equilibrium capital-labour ratio is higher in a non-monetary economy than in a monetary economy; (b) the steady state capital-labour ratio can be affected by changing the rate of growth of the nominal quanity of money. These results indicate that money is not neutral in the sense mentioned above. In Tobin's model, disposable income plays an important role because money affects the real variables through its effect on real disposable income which itself determines savings. Disposable income, Yd, in Tobin's model is:

$$Yd = Y + \frac{d}{dt}\left(\frac{M}{P}\right) , \tag{2.11}$$

which can be written as

$$Yd = Y + \frac{M}{P}(\theta - \pi) , \tag{2.12}$$

where $\theta = \frac{\dot{M}}{M}$, and $\pi = \frac{\dot{P}}{P}$. It is assumed that the expected rate of change in price, π, is equal to the actual rate of change, $\frac{\dot{P}}{P}$. Obviously, the importance of this distinction arises when the two are not the same. We will later consider the case when $\frac{\dot{P}}{P} \neq \pi$. If the government real deficit is equal to $\frac{M}{P}(\theta - \pi)$, we can say that (in real terms) savings equals investment plus the government deficit. Given the assumption of saving as a constant proportion of disposable income, it follows that

$$s\left[Y + \frac{d}{dt}\left(\frac{M}{P}\right)\right] = \dot{K} + \frac{d}{dt}\left(\frac{M}{P}\right) , \tag{2.13}$$

$$\dot{K} = s[Y + \frac{d}{dt}(\frac{M}{P})] - \frac{d}{dt}(\frac{M}{P}) . \quad (2.14)$$

Eq (2.14) is often referred to as "Tobin's fundamental equation." Using the assumption that the labour force grows at some given rate n, Eq (2.14) can be written as

$$\dot{k} = sf(k) - (1 - s)(\theta - \pi)m - nk , \quad (2.15)$$

where $m = \frac{M}{PL}$.

Levharvi and Patinkin challenged both of Tobin's results.[1] In their approach, the opportunity cost of holding real balances is the yield on capital less the yield on real balances. The rate of return on capital is defined to be $\rho = f'(k)$. The rate of return on real money balances is $-\frac{\dot{P}}{P}$. If there is a positive rate of interest on money, r, it becomes $r - \frac{\dot{P}}{P}$. Hence, the opportunity cost of holding real cash balances is $(\rho + \frac{\dot{P}}{P} - r)$, which is equal to the marginal utility of real balances as a consumer good; and $(\rho + \frac{\dot{P}}{P} - r) \frac{M}{P}$ is the real value of the services of real balances. Disposable income in their model is equal to output plus real transfers from government plus the real value of the services of real balances, i.e.,

$$Yd = Y + \frac{M}{P}(\theta - \pi) + \frac{M}{P}(\rho + \pi - r) , \quad (2.16)$$

where $\pi = \frac{\dot{P}}{P}$ is again assumed. When $r = 0$, Eq (2.16) becomes

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[1]D. Levhari and D. Patinkin, "The Role of Money in a Simple Growth Model," American Economic Review 68 (September 1968): 713-53.

(2.17) $$Yd = \frac{M}{P}(\theta + \rho) + Y \ .$$

Eq (2.17) can be written in terms of disposable income per capita:

(2.18) $$yd = f(k) + m(\theta + \rho) \ ,$$

and the physical consumption per capita is

(2.19) $$\frac{C}{L} = (1 - s)yd - m(\rho + \frac{\dot{P}}{P}) \ ,$$

(2.20) $$\frac{C}{L} = (1 - s)f(k) + (1 - s)(\theta + \rho)m - m(\rho + \frac{\dot{P}}{P}) \ .$$

At the steady state we have $\frac{\dot{P}}{P} = \pi = \theta - n$. Therefore,

(2.21) $$\frac{C}{L} = (1 - s)f(k) + (1 - s)(\theta + \rho)m - m(\rho + \theta - n) \ .$$

If we assume that $n = 0$, Eq (2.21) becomes

(2.22) $$\frac{C}{L} = (1 - s)f(k) - s(\rho + \theta)m \ .$$

Since $s > 0$, there exists a negative relationship between per capita real balances, m, and the per capita consumption of goods. Thus, when the services of real balances are regarded as a component of disposable income which yields utility directly, a total consumption is assumed to be proportional to disposable income, we get a negative relation between per capita real balances, m, and the per capita consumption of goods.

In Tobin's model, all markets are cleared at any moment of time. Tobin was interested in the steady state path of an economy and did not show how this steady state is reached. Sidrauski

uses a variant of Tobin's model, and considers the case in which $\frac{\dot{P}}{P} \neq \pi$.[1] He shows that given the rate of monetary expansion, an equilibrium growth path exists, and there is only one stock of capital and one stock of real cash balances associated with an increase in the rate of monetary expansion that will increase the long-run capital-labour ratio. Sidrauski uses the adaptive expectation hypothesis and then derives the per capita equilibrium values of money blances, m^*, and capital, k^*. His model will approach its steady state. If $1 + \gamma\frac{\eta_m}{r} > 0$,[2] a constant rate of monetary expansion will guarantee the stability of the equilibrium growth path without fluctuations in capital stock or in the rate of change in prices. Alternatively, if this condition is not satisfied, the steady state solution to the model is unstable. He says that "... the more rapidly do people adjust their expectation, namely, the higher is the expectation coefficient, the higher will be the probability of the system being unstable.[3]" This is equivalent to saying that the lower the expectation coefficient, the higher the probability of the system being stable.

Hadjimichalakis refers to Tobin's (1965) article and points out that "a careful study of Tobin's paper suggests that he

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[1] M. Sidrauski, "Inflation and Economic Growth," Journal of Political Economy 75 (December 1967): 796-810.

[2] In Sidrauski's terminology γ is the expectation coefficient, r is the opportunity cost of holding real cash balances, $r = \rho + \pi$, and η_m is the demand elasticity of money with respect to r. In Chapter III we will assume that $r = \rho + \pi$ be the market rate of interest on government bonds.

[3] Ibid., p. 807.

does not restrict his analysis to the case of $\frac{\dot{P}}{P} = \pi$.[1]" Hadjimichalakis introduces disequilibrium as well as expectation into the neoclassical monetary growth model and calls it a "Generalized Tobin Model." Since Tobin was interested in the steady state equilibrium solution, he held that the expected and actual rate of inflation were equal, and that the demand for real cash balances was equal to the supply, $\frac{M}{P}$. For this reason, Hadjimichalaki claims that the orginal Tobin model is a special case of the "Generalized Tobin Model." With reference to Tobin's model and the question of whether or not markets can be cleared, Hadjimichalakis shows that the "Generalized Tobin Model" is unstable in both the short-run and the long-run when people adjust their expectation quickly and the speed of adjustment in asset markets is high. Thus, when people adjust their expectation quickly, both Sidrauski and Hadjimichalakis have found that the long-run equilibrium is unstable. We shall attempt to show that, under a certain set of assumptions, the neoclassical monetary growth model with government bonds has a tendency towards stability when the expectation coefficient and the speed of adjustment in asset markets are high. We shall also demonstrate that the equilibrium value of m^* is lower in the neoclassical growth model with government bonds than in the "Generalized Tobin Model." This result is expected since there exists an alternative asset to real capital and money, namely, government

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[1] M. G. Hadjimichalakis, "Equilibrium and Disequilibrium Growth With Money — Tobin Model," Review of Economic Studies 38 (October 1971): 459, footnote 1. Tobin's statements is that "it will take time for the new rate of deflation to register in expectations and for wealth owners to try to adjust to new expectation."

bonds.

The Role of Bonds in a Growing Economy

> *A theory is usually regarded as a monetary theory if the economic system envisaged is one in which the equilibrium interest rate, or the equilibrium pattern of rates, can be altered by a change in the quantity of money.*[1]

Metzler distinguished between two different types of increase or decrease in the quantity of money. The first is the change in the quanity of money which takes place through open-market operations; the second is a direct decrease or increase in the money supply. In the latter case, the money supply is changed without changing private nominal holdings of other assets, but, in the former case, the change in money supply is accompanied by a change in the private holdings of other assets. Metzler contends that the classical theory of the interest rate is a theory of real interest rate in which neither of these two types of change in the money supply can have any effect on the level of the equilibrium rate. The equilibrium rate of interest is independent of both the quantity of money and the policy of the central bank. Therefore, the classical theory is a non-monetary theory. Keynes' theory is a purely monetary theory of interest rate. With respect to the rate of interest, the neo-classical theory has an intermediate position between the classical and the Keynesian theories.

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[1] L. A. Metzler, "Wealth, Saving, and the Rate of Interest," Journal of Political Economy 59 (April 1951): 96.

According to Metzler, when we study monetary economies we should not ignore the interest rate and the role played by the interest rate. But the neoclassical monetary growth model simply ignores the fact that any open-market operation, by the purchasing or selling of securities, will change the quantity of money as well as the interest rate. The neoclassical monetary growth model does not take into account that part of disposable income which is generated by interest payment. Obviously a more complete growth model would have both types of government debt, interest-bearing and non-interest-bearing. However, Metzler introduces the interest rate and bonds into his dynamic model. Since a fall in the real value of money balances is equivalent to an increase in prices, a change in prices can be expressed in terms of a change in the value of money if there is no new borrowing or lending by the banking system.[1] Therefore,

$$\frac{d}{dt}\left(\frac{M}{P}\right) = k_1[S(r,W) - I(r)] \ , \tag{2.23}$$

where k_1, the speed of the price movement, is constant and greater than zero, and saving is assumed to be a function of the rate of interest, r, and the real value of all privately held wealth, W, including both money and securities.

It is assumed that security prices tend to rise whenever asset holders on balance attempt to shift from money to securities

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[1] Ibid., p. 115.

and that they fall when asset holders attempt to shift in opposite direction. The attempted shift, in turn, depends upon whether the actual ratio of cash to securities is higher or lower than the desired ratio. Since a rise in the price of securities is equivalent to a fall in the rate of interest, we can have

$$\dot{r} = k_2\left[L(r) - \frac{(M/P)}{\beta a}\right] , \tag{2.24}$$

where k_2 is a constant, greater than zero, $\frac{M}{P}$ is the real value of private money holdings, a is the real value of all securities (whether held by private owners or by the central bank), β is the proportion of the total supply of securities held by private owners as opposed to that held by the central bank, and $L(r)$ is the real liquidity-preference function. In addition, the following two definitional equations are assumed to be satisfied at any moment of time, without lag:

$$W = a + \frac{M}{P} \tag{2.25}$$

$$a = \frac{cY}{r} \tag{2.26}$$

where Y denotes real national income under the condition of full employment, and cY is business profits.

These four equations, Eqs (2.23) to (2.26), complete Metzler's comparative dynamic system. Taking the linear approximation at a stationary point (r^*, W^*, M^*, and a^*), it has been found that the dynamic model is stable. If the change in prices does not alter the form of the liquidity-preference function, the saving

function, and the investment function, the dynamic system will eventually reach a stationary or static position.

The central bank can affect the rate of growth of the economy by changing the rate of interest. "By purchasing securities, it can reduce the value of private wealth, thereby increasing the propensity to save and causing the system to attain a new equilibrium at a permanently higher rate of capital accumulation. In a similar way, by selling securities, it can increase the real value of private wealth, lower the propensity to save, raise the equilibrium rate of interest, and reduce the rate of capital accumulation.[1]" Obviously, the substantial influence of the bank upon the rate of growth of the economy depends upon the magnitude of the saving-wealth relation. If the saving-wealth relation is large, the propensity to save increases or decreases as the real value of private wealth falls or rises, if the bank sells or buys securities in large quantities, the rate of growth may be affected to a considerable extent by central bank policy. In particular, of course, there is a limit to the amount of securities the bank can purchase and sell; it cannot sell more than it owns or buy more securities than are available in the system.

As Metzler himself mentioned, this dynamic system made no allowance for expectation, and expectation may have a destabilizing influence on the system. A rise in commodity prices may lead people to anticipate a further price increase. This will probably cause

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[1] Ibid., p. 112.

saving to decline and investment to increase, thereby widening the inflationary gap and causing further price increases. Likewise, if security prices are rising, asset holders may anticipate a further price increase, and attempt to shift from money to securities. This will cause a further rise in security prices.

Enthoven introduces bonds into the dynamic model.[1] He assumes that investment, business saving, and households' demand for bonds and consumption are all homogeneous of degree one in the asset and income variables, but not in the price level. In his model, partial analysis is used to show the stability of factor proportions and the ratio of debt to income. Enthoven is able to show the stability of the capital-labour ratio for his model in balanced growth.[2] The interest rate is negatively related to investment. Higher level of interest rate will correspond to lower level of the investment function. In addition, Enthoven shows that as debt and income grow, the ratio of debt to income will converge to a stable, limiting value[3] and illustrates the stability of the nominal bonds outstanding-capital ratio. His model, like that by Metzler, makes no allowance

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[1] A. G. Enthoven, "A Neoclassical Model of Money, Debt and Economic Growth," the Mathematical Appendix to J. G. Gurley and E.S. Shaw, Money in a Theory of Finance (Washington, D.C.: The Brookings Institution, 1960): 303-58.

[2] The investment function is a decreasing function of the capital-labour ratio.

[3] A limiting value will be greater, the greater is the effect of income on borrowing, the lower is the rate of growth of income and the less is the deterrent effect of existing debt on new borrowing.

for expectations. Moreover, it assumes that the system has an equilibrium solution for each variable; then the effect of change in each variable is examined.

Ethier introduces bonds and the interest rate into both the neoclassical and the Keynesian monetary growth models.[1] His short-run model of the neoclassical type shows that an increase in the rate of growth of financial assets will raise the price level, reduce the real rate of interest, and raise the rate of capital accumulation. He does not show the conditions under which the short-run equilibrium is stable; nor does he show whether or not there exists a limit or a range for the ratio of the government debts within which the stability conditions are satisfied. In his long-run dynamic model, he emphasized the effects of the long-run values of the rate of growth of financial assets and the government debt ratio on the nature of the long-run equilibrium, rather than the conditions under which the long-run equilibrium is stable.

In the following chapters, our purpose is to introduce government bonds into the neoclassical monetary growth model and to examine the stability properties, if any, for both the short-run and long-run dynamic system.

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[1] W. Ethier, "Financial Assets and Economic Growth in a Keynesian Economy," Journal of Money, Credit, and Banking 7 (May 1975): 213-32.

CHAPTER III

DESCRIPTION OF THE MODEL

The economy is divided into three sections: business firms, households, and the government. Following the neoclassical growth model, we assume that there exists a single output, Y, used both as capital and as consumer good, which is produced by two factors of production, namely, capital, K, and labour, L,

$$Y = F(K,L) .$$

We assume a well-behaved production function. By "well-behaved," we mean the function F is twice continuously differentiable, linearly homogeneous and, in addition, has the following properties: (1) positive and diminishing marginal productivities of the two factors, and (2) zero output at a zero input level, (Impossibility of the Land of Cockaigne).[1]

$$y = f(k) > 0, \quad \text{for} \quad 0 < k < \infty$$

$$f'(k) > 0, \quad f''(k) < 0, \quad \text{for} \quad k > 0$$

$$\lim_{k \to 0} f'(k) = \infty , \qquad \lim_{k \to \infty} f'(k) = 0$$

and, in addition

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[1] T. C. Koopmans, "Analysis of Production as an Efficient Combination of Activities," in Activity Analysis of Production and Allocation (New York: John Wiley, 1951): 49.

$$f(0) = 0, \qquad \lim_{k \to \infty} f(k) = \infty ,$$

where $y = \frac{Y}{L}$, and $k = \frac{K}{L}$.

Households supply labour and capital to the factor markets. The factor markets are assumed always to be in competitive equilibrium, and each factor is paid the value of its marginal product. Therefore, the real rent on capital, ρ, is always equal to the marginal product of capital; the real wage, w, is always equal to the marginal product of labour. Firms employ these factors of production in order to produce the single output Y. Households receive wages, income from assets, and transfer payments from the government sector. They save a constant proportion s (where $0 < s < 1$) of their disposable income. They determine how their savings should be held among the available assets. The government sector can run a budgetary deficit, financed by issues of new money and bonds, in order to buy a part of the current output of goods, to buy outstanding bonds, to make transfer payments or to make interest payments to bond holders. We assume that there are no commercial banks or private bonds in the economy. Therefore, the available assets are real capital, government non-interest-bearing debt (money), and government interest-bearing debt (bonds). With money, bonds, and real capital, and a single output Y which can either be consumed, or saved and invested, let us call this model a "Three asset one-sector growth model."

The rate of return on real capital is ρ, which is equal to the marginal product of capital. The rate of return on money and bonds depends on the interest rate and the price level. The rate of

return on money is equal to the negative of the changing price level, and the rate of return on bonds is equal to the market interest rate on government bonds, r, plus the negative of the changing price level.

The total government expenditures must be equal to the total government financing from all available sources. In other words, government revenue from taxation, V, funds borrowed from private lenders in exchange for bonds, $\frac{\dot{B}}{rP}$, and new issue of money, $\frac{\dot{M}}{P}$, must be equal to transfer payments, $\frac{V^o}{P}$, interest payments to bond holders, $\frac{B}{P}$, and purchases of goods and services, G, i.e.,

$$V + \frac{\dot{B}}{rP} + \frac{\dot{M}}{P} = \frac{V^o}{P} + \frac{B}{P} + G$$

where $V = U(Y + \frac{B}{P})$; U is the tax rate.[1]

Let us assume that the government expenditures on goods and services are equal to the government revenue from taxation, i.e., $G = V$. Therefore, $\frac{\dot{B}}{rP} + \frac{\dot{M}}{P} = \frac{V^o}{P} + \frac{B}{P}$. Thus, the government deficits are entirely financed by the creation of government non-interest-bearing debts, money, and government interest-bearing debts, bonds.

Since the model includes money and bonds, transfer payments, and interest payments by the government, disposable income will be equal to the total payments to the factors of production, capital and labour, plus the interest payment on the outstanding bonds and the

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[1]This equation is from T. Tsushima, "On a Dynamic Macroeconomic Model for Monetary and Fiscal Policy," Unpublished paper presented at the CEA Meeting, Toronto, June 1974.

transfer payments:

$$(3.1) \qquad Yd = F(K,L) + \frac{d}{dt}\left(\frac{M}{P}\right) + \frac{d}{dt}\left(\frac{B}{rP}\right) .$$

Here Yd is disposable income, B is the number of nominal bonds outstanding, M is the nominal stock of money, r is the interest rate on bonds, and P is the price level of goods. Eq (3.1) can be written in terms of per capita disposable income:

$$(3.2) \qquad yd = f(k) + m\left(\frac{\dot{M}}{M} - \frac{\dot{P}}{P}\right) + b\left(\frac{\dot{B}}{B} - \frac{\dot{P}}{P} - \frac{\dot{r}}{r}\right) ,$$

where $m = \frac{M}{PL}$, $b = \frac{B}{rPL}$, and the dot over B, M, P, and r denotes the first derivative of the corresponding variables with respect to time.

If the rate of return on money, bonds, and capital is such that households are willing to hold less money and bonds, capital intensity will increase and the rate of return on capital will fall. Since the rate of return on capital must be greater than zero, the properties of a "well-behaved" production function imply that there exists a limit for capital intensity. At the stationary point $\frac{\dot{k}}{k} = 0$, i.e., the real sector achieves the steady state growth path, provided that the rate of interest remains unchanged. Given the rate of return on capital, in order to have a constant rate of interest the authorities must expand money and bonds at the same rate (justification for this statement will be presented later). We assume $r = \rho + \pi$, the nominal rate of interest is the sum of ρ and π. Given r and ρ, the expected rate of inflation becomes constant. On the

equilibrium growth path the expected rate of inflation is equal to the rate of money expansion minus the natural rate of growth, n. At the steady state equilibrium path, the ratio of the government debts, θ_1, must be constant, i.e.,

$$\frac{(M/P)}{(B/rP)} = \frac{(M/PL)}{(B/rPL)} = \frac{m}{b} = \theta_1 .$$

By logarithmic differentiation of this equation with respect to time, we get

$$m(\frac{\dot{M}}{M} - \frac{\dot{P}}{P} - n) = b(\frac{\dot{B}}{B} - \frac{\dot{P}}{P} - \frac{\dot{r}}{r} - n) = 0 ,$$

where the labour force is assumed to grow at a constant rate n. By substituting $m = b\theta_1$ we get

$$b\theta_1(\frac{\dot{M}}{M} - \frac{\dot{P}}{P} - n) - b(\frac{\dot{B}}{B} - \frac{\dot{P}}{P} - \frac{\dot{r}}{r} - n) = 0 ,$$

$$\theta_1 \frac{\dot{M}}{M} - \frac{\dot{B}}{B} - (\theta_1 - 1)(\pi + n) = 0 ,$$

since at the steady state growth path $\dot{r} = 0$, and $\pi = \frac{\dot{P}}{P}$. This equality holds at the steady state when $\frac{\dot{M}}{M} = \frac{\dot{B}}{B} = \theta$, $(\theta_1 - 1)(\theta - \pi - n) = 0$, for $(\theta_1 - 1) \neq 0$, i.e., $\pi^* = \theta - n$, which indicates that on the equilibrium growth path, the expected rate of inflation is equal to the rate of monetary expansion minus the natural rate of growth of an economy.

By substituting $m = b\theta_1$, $\frac{\dot{M}}{M} = \frac{\dot{B}}{B} = \theta$, and the expected rate of change of price level π, into Eq (3.2), we get

(3.3) $$y = f(k) + (\theta - \pi)(1 + \theta_1)b - \frac{\dot{r}}{r} b ,$$

and saving per capita becomes

(3.4) $$\frac{S}{L} = sy = sf(k) + s[(\theta - \pi)(1 + \theta_1)b - \frac{\dot{r}}{r} b] .$$

This per capita saving must equal the net addition to the community's per capita stock of wealth:

(3.5) $$\frac{S}{L} = \frac{\dot{K}}{L} + \frac{1}{L}\frac{d}{dt}\left(\frac{M}{P} + \frac{B}{rP}\right) ,$$

or equivalently,

$$\frac{S}{L} = \dot{k} + nk + (\theta - \pi)(1 + \theta_1)b - \frac{\dot{r}}{r} b .$$

Therefore,

(3.6) $$\dot{k} = sf(k) - (1 - s)[(\theta - \pi)(1 + \theta_1)b - \frac{\dot{r}}{r} b] - nk .$$

At steady state equilibrium, the expected and actual rate of inflation are equal, the capital-labour ratio and the rate of interest are fixed, and money and bonds grow at the same rate. But, before an economy reaches the steady state growth path, it is possible that the various markets may not be in equilibrium and that π may differ from $\frac{\dot{P}}{P}$. In fact, whether or not an economy can reach an equilibrium growth path is a question to be answered. For this reason, we allow the possibility of disequilibrium in this model and then show that the economy does indeed reach a steady state equilibrium under certain assumptions. We assume that the labour market is at equilibrium at any moment of time, that the money

and bond markets are not at equilibrium, and that the expected rate of change in the price level is different from the actual rate of change. According to Walras' law, the positive (negative) excess demand for goods and services will cause the price level to rise (fall). Since it is assumed that the labour market is at equilibrium, the price level must change in order to ensure the equilibrium in the commodity market. Thus, the actual rate of change of inflation can be written as

$$\frac{\dot{P}}{P} = \varepsilon[m + b - H(\cdot) - \hat{B}(\cdot)] \,, \tag{3.7}$$

where $H(\cdot)$ and $\hat{B}(\cdot)$ are the demand functions for money and bonds respectively (they will be specified later), and ε is the speed of adjustment which is constant and greater than zero. If $\varepsilon \rightarrow \infty$ (or $\frac{1}{\varepsilon} \rightarrow 0$), the sum of excess demands in the money and bond market becomes zero.

For the dynamic behaviour of expectations, we use the so-called "adaptive expectation" equation that was originally introduced by Cagan (1956)[1] and used by Sidrauski (1967), and Hadjimichalakis (1971) in their monetary growth models. Suppose that the quantity demanded (as in a period of rapid inflation) is determined by the expected price level rather than the actual price level. Since the expected price is not directly observable, we postulate that the value of expectations is formed on the adaptive rule

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[1] P. Cagan, "The Monetary Dynamics of Hyperinflation," in M. Friedman, ed., Studies in the Quantity Theory of Money (Chicago: University of Chicago Press, 1956), p. 37.

$$(3.8) \qquad \dot{\pi}_t = \pi_t - \pi_{t-1} = \gamma[(\frac{\dot{P}}{P})_t - \pi_{t-1}] ,$$

where t denotes the time period. γ is a constant, greater than zero, and is called the "expectation coefficient." This means that current expectations are formed by adapting previous expectations in the light of actual achievements, i.e., current experience. Expectations are reformulated in each period; $\pi_t - \pi_{t-1}$ is the change in current expectations. The change is only a fraction of the difference between the realized rate of inflation, $(\frac{\dot{P}}{P})_t$, and the previously expected rate of inflation, π_{t-1}. For the period $t+i$ we have

$$(3.9) \qquad \dot{\pi}_{t+i} = \gamma[(\frac{\dot{P}}{P})_{t+i} - \pi_{t+i-1}] ,$$

and for a large value of i we have

$$(3.10) \qquad \dot{\pi} = \gamma[\frac{\dot{P}}{P} - \pi] .$$

If the expected rate of inflation is less than the actual one, the expected rate of inflation rises, and vice versa. If $\gamma \to \infty$ (or $\frac{1}{\gamma} \to 0$, $\frac{1}{\gamma}$ being the time period needed for correcting the inequality between $\frac{\dot{P}}{P}$ and π) then we have $\frac{\dot{P}}{P} = \pi$, the case of "perfect-foresight."

We next specify the demand functions for money and bonds, $H(\cdot)$ and $\hat{B}(\cdot)$. Money is demanded for two purposes: to finance transactions and to hold as an asset. The per capita demand for money as a means of payment is a function of per capita output, and the demand to satisfy precautionary, as well as speculative, motives is a function of the opportunity cost of holding these balances,

which, in turn, depends on the rental on capital, ρ, the expected rate of return on cash balances, and the rate of return on bonds. Therefore, combining these we get

$$(3.11) \qquad m = H[r, \rho, \pi, f(k)] .$$

It is assumed that there exist uncertainties as to the timing of payments which cause households to hold assets in the form of money, bonds, and capital. In deciding on the composition of their asset portfolios, households are actually concerned with the anticipated rates of return from these assets. We adopt the assumption of $\pi \neq \frac{\dot{P}}{P}$. We also assume that the anticipated rate of interest is equal to the observed market rate on the government bonds, r, consisting of

$$(3.12) \qquad r = \rho + \pi ,$$

which is Fisher's money rate of interest.

Since the demand for money is an increasing function of $f(k)$, and the latter, in turn, is an increasing function of k, the demand for money becomes an increasing function of the capital-labour ratio. In addition, an increase in the capital-labour ratio means a fall in the marginal product of capital, i.e., ρ, which lowers the opportunity cost of money. The opportunity cost of holding real cash balances can be written as

$$\max. (\rho, r) + \pi ,$$

the difference between the rental on capital, or the difference

between the market rate of interest on government bonds, whichever is greater, and the (expected) real yield of real cash balances, $-\pi$. By substituting $\pi = r - \rho$, we get

$$\max. (\rho, r) + r - \rho ,$$

which is a function of k and r. The demand for real cash balances is an inverse function of the opportunity cost of holding real cash balances. Therefore, the demand equation for money can be written as

$$m = H(k, r), \quad \text{where} \quad H_k > 0, \ H_r < 0. \tag{3.13}$$

The real value of bond holdings demanded depends on real income, the rate of interest, the rate of return on capital, and the expected rate of return on real cash balances

$$b = \hat{B}[r, \rho, \pi, f(k)] \ . \tag{3.14}$$

The higher the output per capita, the higher will be saving per capita and, thus, more bonds will be demanded. The higher output per capita implies the higher capital-labour ratio. Therefore, the demand for bonds is an increasing function of capital-labour ratio. A fall in the rate of interest (i.e., a rise in the price of bonds) leads to an increase in the opportunity cost of holding bonds, and lowers the demand for bonds. A rise in the rate of interest (i.e., a fall in the price of bonds) has an opposite effect on the demand for bonds. Therefore, the demand for bonds is an increasing function of the rate of interest. The opportunity cost of holding government bonds can be written as

$$\max.\ (\rho, -\pi) - r ,$$

the difference between the rental on capital, or the difference between the (expected) real yield of real cash balances, $-\pi$, whichever is greater, and the market rate of interest on government bonds. By substituting $\pi = r - \rho$, we get

$$\max.\ (\rho, \rho - r) - r = \rho - r \quad \text{for} \quad r \geqq 0 ,$$

which is a function of k and r. Therefore, the demand equation for bonds can be written as

(3.15) $$b = \hat{B}(k, r), \quad \text{where} \quad \hat{B}_k > 0, \quad \hat{B}_r > 0 .$$

Substituting (3.13) and (3.15) into (3.7), and remembering that $m = b\theta_1$, we get

(3.16) $$\frac{\dot{P}}{P} = \varepsilon[(1 + \theta_1)b - H(k,r) - \hat{B}(k,r)] .$$

Differentiating logarithmically $m = \frac{M}{PL}$, $b = \frac{B}{rPL}$, and $r = \rho + \pi$ with respect to time we get

(3.17) $$\dot{m} = m(\theta - \frac{\dot{P}}{P} - n) ,$$

(3.18) $$\dot{b} = b(\theta - \frac{\dot{P}}{P} - \frac{\dot{r}}{r} - n) ,$$

(3.19) $$\dot{r} = f''\dot{k} + \dot{\pi} .$$

By collecting Eqs (3.6), (3.10), (3.16), (3.17), (3.18), and (3.19), our model may be represented by the following system of six differential equations:

$$(3.20)\qquad \begin{cases} \dot{k} = sf(k) - (1-s)[(\theta - \pi)(1+\theta_1)b - \frac{\dot{r}}{r}b] - nk \ , \\[2ex] \dot{m} = m(\theta - \frac{\dot{P}}{P} - n) \ , \\[2ex] \dot{b} = b(\theta - \frac{\dot{P}}{P} - \frac{\dot{r}}{r} - n) \ , \\[2ex] \frac{\dot{P}}{P} = \varepsilon[(1+\theta)b - H(k,r) - \hat{B}(k,r)] \ , \\[2ex] \dot{r} = f''\dot{k} - \dot{\pi} \ , \\[2ex] \dot{\pi} = \gamma[\frac{\dot{P}}{P} - \pi] \ . \end{cases}$$

Since $m = b\theta_1$, we can eliminate Eq (3.17), and by substituting $\frac{\dot{P}}{P}$ into the $\dot{\pi}$ and $\dot{b}$ equations and $\dot{\pi}$ into $\dot{r}$ equation, our three asset one-sector model may be reduced to the following system of three differential equations in $\dot{k}$, $\dot{b}$, and $\dot{r}$ only:

$$(3.21)\qquad \begin{cases} \dot{k} = sf(k) - (1-s)[(\theta - \pi)(1+\theta_1)b - \frac{\dot{r}}{r}b] - nk \ , \\[2ex] \dot{b} = b\{\theta - \varepsilon[(1+\theta_1)b - H(k,r) - \hat{B}(k,r)] - \frac{\dot{r}}{r} - n\} \ , \\[2ex] \dot{r} = f''\dot{k} + \gamma\{\varepsilon[(1+\theta_1)b - H(k,r) - \hat{B}(k,r)] - \pi\} \ . \end{cases}$$

The stock variables k and b, and the market interest rate, r, must change so as to clear each market involved as an economy evolves over time according to the above three equation system.

CHAPTER IV

AN EXAMINATION OF STABILITY PROPERTIES

Short-run Analysis

In this section, we attempt to examine whether the short-run equilibrium exists, and under what condition(s) an economy can reach an equilibrium in the short-run. For this purpose we consider the short-run analysis of the system described by Eqs (3.21). The short-run equilibrium could be defined as a temporary equilibrium for a time period in which the capital intensity is fixed at $\bar{k}$ and the labour force is given, so that $\dot{k} = n = 0$. This implies that the marginal product of capital (the real rate of return on capital) will remain unchanged. We note that k can be constant if the supply of labour is equal to the demand for labour. But by our assumption that the labour market is in equilibrium at any moment of time, $\bar{k}$ can be taken as constant in the short-run period. Therefore, setting $\dot{k} = n = 0$ in Eq (3.21), our short-run system is reduced to two equations system for b and r:

$$(4.1) \qquad \begin{cases} \dot{b} = b\{\theta - \varepsilon[(1+\theta_1)b - H(\bar{k},r) - \hat{B}(\bar{k},r)] - \frac{\dot{r}}{r}\}, \\ \\ \dot{r} = \gamma\{\varepsilon[(1+\theta_1)b - H(\bar{k},r) - \hat{B}(\bar{k},r)] - \pi\}. \end{cases}$$

For the short-run equilibrium, we consider the local stability of the model. The stationary point $(\bar{b},\bar{r})$ can be found by setting $\dot{b} = \dot{r} = 0$. We then get

$$\theta = \bar{\pi} = \frac{\dot{P}}{P} = \varepsilon[(1 + \theta_1)\bar{b} - H(\bar{k},\bar{r}) - \hat{B}(\bar{k},\bar{r})] \quad . \tag{4.2}$$

The stationary point requires that the rates of money and bond expansion, θ, be equal to the expected rate of inflation, $\bar{\pi}$, which, in turn, must be equal to the actual rate of inflation, $\frac{\dot{P}}{P}$. One might ask what will happen if the rate of change in money differs from the rate of change in bonds. In order to answer this question, let us assume that the following relations hold:[1]

$$\begin{cases} m = H(k,r) \\ b = \hat{B}(k,r) \ . \end{cases} \tag{4.3}$$

Taking the time derivatives of these equations, setting $\dot{k} = n = 0$, and solving simultaneously for $\frac{\dot{r}}{r}$ and $\frac{\dot{P}}{P}$ we get

$$\frac{\dot{r}}{r} = \frac{\frac{\dot{M}}{M} - \frac{\dot{B}}{B}}{\eta_m - (1 + \eta_b)} \gtreqless 0 \tag{4.4}$$

$$\frac{\dot{P}}{P} = \frac{\eta_m \frac{\dot{B}}{B} - (1 + \eta_b) \frac{\dot{M}}{M}}{\eta_m - (1 + \eta_b)} > 0 \ , \tag{4.5}$$

where η_m and η_b denote the demand elasticities for money and bonds with respect to the rate of interest, $\eta_m < 0$, and $\eta_b > 0$. Eq (4.5) shows that the price level is always increasing so long as there exists a positive growth rate of money and bonds. Eq (4.4) suggests that the rate of change in the market rate of

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[1]This section draws heavily on Tsushima's results of short-run analysis in "On a Dynamic Macroeconomic Model for Monetary and Fiscal Policy."

interest is a function of the difference between the rate of change in money and the rate of change in bonds. For example, when the rate of change in money is less than that of bonds, the market rate of interest rises. Suppose that real money balances per capita, m, are less than desired at equilibrium, $m < m^*$. The authorities must then increase the rate of issue of money, with an open-market purchase of bonds. This causes the interest rate to decline, and is accompanied by a rising price level. Since the market rate of interest decline and the price level rises, this may leave the real value of the bond holdings unchanged. This process will continue until the desired ratio of m^* and b^* is achieved. If the authorities wish to hold the price level unchanged, the following relation between $\frac{\dot{B}}{B}$ and $\frac{\dot{M}}{M}$ must hold by putting $\frac{\dot{P}}{P} = 0$ in Eq (4.5):

$$\frac{\dot{B}}{B} = \frac{1 + \eta_b}{\eta_m} \frac{\dot{M}}{M} , \tag{4.6}$$

where $\frac{\dot{B}}{B}$ and $\frac{\dot{M}}{M}$ are not zero. Since $\frac{1 + \eta_b}{\eta_m} < 0$, this implies that an open-market purchase of bonds should take place when $\frac{\dot{M}}{M} > 0$. The market rate of interest would go down, causing the real value of the bond holding in the private sector to rise until the desired ratio between m^* and b^* is again achieved.

By taking the Taylor linear approximation at a stationary point $(\bar{b}, \bar{r})$, we get the following matrix of the short-run dynamic model (4.1)

$$(4.7)\quad \begin{bmatrix} -\varepsilon b(1+\theta_1)(1+\frac{\gamma}{r}) & b\{(1+\frac{\gamma}{r})[\varepsilon\{(1+\theta_1)\frac{b}{r}+H_r+\hat{B}_r\}]+\frac{\dot{r}}{r}\} \\ \varepsilon\gamma(1+\theta_1) & -\gamma\{\varepsilon[(1+\theta_1)\frac{b}{r}+H_r+\hat{B}_r]+1\} \end{bmatrix},$$

where an element of the matrix, a_{ij}, stands for partial derivative of i^{th} equation with respect to j^{th} variable. In addition, it should be pointed out here that the stability analysis is based on the behaviour of the system of differential equations in the neighborhood of a stationary point, i.e., "local" in nature, due to a linear approximation. A set of necessary and sufficient conditions for local stability of the system is that the trace of (4.7) be negative and the determinant of (4.7) be positive.[1] From (4.7) we get

$$\text{det.} = b(1+\theta_1) > 0$$

$$\text{trace} = -\{\varepsilon b(1+\theta_1)(1+\frac{\gamma}{r}) + \gamma\varepsilon[(1+\theta_1)\frac{b}{r}+H_r+\hat{B}_r]+\gamma\}$$

$$= -\varepsilon\gamma[b(1+\theta_1)(\frac{1}{\gamma}+\frac{2}{r})+H_r+\hat{B}_r+\frac{1}{\varepsilon}] .$$

Since the determinant is always positive and $H_r < 0$, $\hat{B}_r > 0$, the necessary and sufficient conditions are satisfied with the following condition(s):

$$\text{if} \quad H_r + \hat{B}_r \geqq 0 ,$$

[1]This criterion is based on the Routh theorem which will be presented on page 42.

$$\text{or} \quad (1 + \theta_1)\frac{b}{r} + H_r + \hat{B}_r > 0 \ ,$$

$$\text{or} \quad \frac{1}{\varepsilon} + H_r + \hat{B}_r > 0 \ ,$$

or the sum of the expression in brackets is greater than zero. If γ and ε both approach infinity, the required condition for stability would be either

$$H_r + \hat{B}_r \geqq 0 \ ,$$

or

$$\frac{2b}{r}(1 + \theta_1) + H_r + \hat{B}_r > 0 \ .$$

If only $\gamma \to \infty$, we need; $\varepsilon[\frac{1}{\varepsilon} + \frac{2b}{r}(1 + \theta_1) + H_r + \hat{B}_r] > 0$, or equivalently,

$$\varepsilon > \frac{-1}{\frac{2b}{r}(1 + \theta_1) + H_r + \hat{B}_r} \ .$$

From the above condition(s), we see that short-run equilibrium can be achieved even though both γ and $\varepsilon \to \infty$, or either $\gamma \to \infty$ or $\varepsilon \to \infty$, if one of the following conditions hold:

$$H_r + \hat{B}_r \geqq 0 \ , \tag{4.8}$$

or

$$\frac{b}{r}(1 + \theta_1) + H_r + \hat{B}_r > 0 \ . \tag{4.9}$$

In order to write the conditions (4.8) and (4.9) in terms of demand elasticities for money and bonds with respect to the rate

of interest, we note that

$$H_r = \eta_m \frac{m}{r} = \eta_m \frac{b}{r} \theta_1 < 0 \ ,$$

$$\hat{B}_r = \eta_b \frac{b}{r} > 0 \ ,$$

where η_m and η_b denote the demand elasticities for money and bonds with respect to the interest rate. Substituting H_r and $\hat{B}_r$ into (4.8) and (4.9), we get

$$(4.8)' \qquad \frac{b}{r}(\eta_b + \eta_m \theta_1) \geqq 0$$

$$(4.9)' \qquad \frac{b}{r}[\eta_b + 1 + \theta_1(\eta_m + 1)] > 0 \ .$$

From (4.8)' we have $\theta_1 \leqq -\frac{\eta_b}{\eta_m}$. This means that the short-run equilibrium is stable if the desired ratio of money and bonds is less than, or equal to, the negative of the ratio of the demand elasticities for bonds and money with respect to the rate of interest. In other words, there exists not only a range for the ratio of the two kinds of government debt within which the short-run equilibrium is stable, but there also exists a limit to this range depending on the magnitude of the demand elasticities for money and bonds with respect to the rate of interest.

If the ratio of the government debt is greater than the negative of the ratio of these demand elasticities, the condition (4.8)' does not hold. Given $\theta_1 \geqq -\frac{\eta_b}{\eta_m}$, (4.9)' indicates two possibilities:

(i) when $0 > \eta_m \geqq -1$, the short-run equilibrium is stable;

(ii) when $\eta_m < -1$, then $\theta_1 < -\dfrac{\eta_b + 1}{\eta_m + 1}$, and there exists a range within which the short-run equilibrium will be stable again --

$$-\frac{\eta_b}{\eta_m} \leqq \theta_1 < -\frac{\eta_b + 1}{\eta_m + 1} .$$

Suppose that (4.8)' holds and the demand elasticity of bonds with respect to the rate of interest is equal to the absolute value of the demand elasticity for money with respect to the rate of interest, i.e., $\eta_b = -\eta_m$, then $\theta_1 \leqq 1$. Alternatively, suppose that (4.8)' does not hold, but (4.9)' does, and $\eta_b = -\eta_m$. Then

$$1 \leqq \theta_1 < \frac{\eta_b + 1}{\eta_b - 1} .$$

The analysis so far has demonstrated that the short-run equilibrium is stable if $\theta_1 \leqq -\dfrac{\eta_b}{\eta_m}$, or $-\dfrac{\eta_b}{\eta_m} \leqq \theta_1 < -\dfrac{\eta_b + 1}{\eta_m + 1}$, no matter what the values of γ, and ε might be.

<u>Long-run Analysis</u>

Although the short-run dynamic model has proved to be stable, it has been assumed that the real sector is fixed and the market rate of interest has no significant effect on the real sector of an economy. In this section, we take the system (3.21) as a long-run dynamic model by allowing $\dot{k}$ to change as the economy evolves over time. We reproduce Eqs (3.21):

$$(4.10)\quad \begin{cases} \dot{k} = sf(k) - (1 - s)[(\theta - \pi)(1 + \theta_1)b - \frac{\dot{r}}{r} b] - nk \ , \\ \dot{b} = b\{\theta - \varepsilon[(1 + \theta_1)b - H(k,r) - \hat{B}(k,r)] - \frac{\dot{r}}{r} - n\} \ , \\ \dot{r} = f''\dot{k} + \gamma\{\varepsilon[(1 + \theta_1)b - H(k,r) - \hat{B}(k,r)] - \pi\} \ . \end{cases}$$

Setting $\dot{k} = \dot{b} = \dot{r} = 0$, we get the long-run equilibrium point (k^*, b^*, r^*), where $*$ denotes equilibrium values.

$$(4.11)\qquad sf(k^*) - n(1 + \theta_1^*)(1 - s)b^* - nk^* = 0 \ ,$$

$$(4.12)\qquad \varepsilon[(1 + \theta_1^*)b^* - H(k^*,r^*) - \hat{B}(k^*,r^*)] = \theta - n \ ,$$

$$(4.13)\qquad \varepsilon[(1 + \theta_1^*)b^* - H(k^*,r^*) - \hat{B}(k^*,r^*)] = \pi^* \ ,$$

where $b \neq 0$. From Eqs (4.11), (4.12), and (4.13), we get the long-run equilibrium values

$$(4.14)\qquad b^* = \frac{sf(k^*) - nk^*}{n(1 - s)(1 + \theta_1^*)} \ ,$$

$$(4.15)\qquad m^* = \frac{sf(k^*) - nk^*}{n(1 - s)(1 + \theta_1^*)/\theta_1^*} \ ,$$

$$(4.16)\qquad \pi^* = \frac{\dot{P}}{P} = \theta - n \ ,$$

$$(4.17)\qquad r^* = f'(k^*) + \theta - n \ .$$

Eqs (4.14) and (4.15) indicate that neither bonds nor money are neutral, in the sense that the capital-labour ratio in our three asset one-sector economy is lower than in a one-sector growth model.

Since $b^* > 0$ and $m^* > 0$, it follows that $sf(k^*) - nk^* > 0$, or $f(k^*)/k^* > n/s$; whereas, in Solow's one-sector growth model, we have $f(k^{**})/k^{**} = n/s$, where $**$ denotes the equilibrium value. Given the properties of the production function, it follows that $k^* < k^{**}$. In comparing Eq (4.15) with the long-run equilibrium value of m^* in Hadjimichalakis' generalized Tobin model where $m^* = \frac{sf(k^*) - nk^*}{n(1 - s)}$,[1] we find that the long-run equilibrium value of m^* in the three asset one-sector model is lower than its value in a two asset one-sector model, since $\frac{1}{\theta_1^*} > 0$, or equivalently $\frac{1 + \theta_1^*}{\theta_1^*} > 1$, $(b \neq 0)$.

By taking the Taylor linear approximation at the long-run equilibrium point (k^*, b^*, r^*) we get the following matrix of the dynamic model:

$$\text{(4.18)} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where

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[1] M. G. Hadjimichalakis, "Equilibrium and Disequilibrium Growth With Money - Tobin Model," Review of Economic Studies 38 (October 1971): 470.

(4.19)

$$a_{11} = \frac{1}{A}\{sf' + (1 - s)[f''(1 + \theta_1)b - \frac{b}{r}\{(H_k + \hat{B}_k) - f''\}] - n\} ,$$

$$a_{21} = \varepsilon b(H_k + \hat{B}_k) - \frac{b}{Ar}\{sf'f'' + f''(1 - s)(1 + \theta_1)b - nf'' - \gamma\varepsilon(H_k + \hat{B}_k) + \gamma f''\} ,$$

$$a_{31} = \frac{1}{A}\{sf'f'' + f''(1 - s)(1 + \theta_1)b - nf'' - \gamma\varepsilon(H_k + \hat{B}_k) + \gamma f''\} ,$$

$$a_{12} = \frac{1}{A}(1 - s)(1 + \theta_1)(\frac{\gamma\varepsilon b}{r} - n) ,$$

$$a_{22} = -\varepsilon b(1 + \theta_1) + \frac{b}{Ar}(1 + \theta_1)[f''(1 - s)n - \gamma\varepsilon] ,$$

$$a_{32} = \frac{1}{A}(1 + \theta_1)[-f''(1 - s)n + \gamma\varepsilon] ,$$

$$a_{13} = \frac{1}{A}(1 - s)\{(1 + \theta_1)b + \frac{nb}{r}(1 + \theta_1) - \frac{\gamma b}{r}\{\varepsilon[\frac{b}{r}(1 + \theta_1) + H_r + \hat{B}_r] + 1\}\} ,$$

$$a_{23} = \varepsilon b[\frac{b}{r}(1 + \theta_1) + H_r + \hat{B}_r] - \frac{b}{Ar}\{f''(1 - s)(1 + \theta_1)(1 + \frac{n}{r})b - \gamma\{\varepsilon[\frac{b}{r}(1 + \theta_1) + H_r + \hat{B}_r] + 1\}\} ,$$

$$a_{33} = \frac{1}{A}\{f''(1 - s)(1 + \theta_1)(1 + \frac{n}{r})b - \gamma\{\varepsilon[\frac{b}{r}(1 + \theta_1) + H_r + \hat{B}_r] + 1\}\} .$$

where $A = 1 - (1 - s)\frac{b}{r}f'' > 0$ for simplicity. In order to make the calculation simple, let us introduce the following notation:

$$D = sf' + f''(1 - s)(1 + \theta_1)b - n \gtreqless ?$$

$$Q = \frac{b}{r}(1 + \theta_1) + H_r + \hat{B}_r > 0 \qquad \text{(the short-run stability condition)}$$

$$J = H_k + \hat{B}_k > 0$$

$$R = \gamma(\varepsilon Q + 1) > 0$$

$$X = \gamma(\varepsilon J - f'') > 0$$

$$Z = (1 + \theta_1)(1 - s)(b + \frac{nb}{r}) > 0 \; .$$

Using this notation, the matrix (4.19) becomes (4.20)

$$\begin{bmatrix} \frac{1}{A}[D - \frac{b}{r}(1 - s)X] & \frac{(1 - s)(1 + \theta_1)(\gamma\varepsilon \frac{b}{r} - n)}{A} & \frac{1}{A}[Z - \frac{b}{r}(1 - s)R] \\ \varepsilon bJ - \frac{b}{rA}(f''D - X) & - \varepsilon b(1 + \theta_1) + \frac{b(1 + \theta_1)}{rA}[f''(1 - s)n - \gamma\varepsilon] & \varepsilon bQ - \frac{b}{rA}(f''Z - R) \\ \frac{1}{A}(f''D - X) & \frac{-(1 + \theta_1)}{A}[f''(1 - s)n - \gamma\varepsilon] & \frac{1}{A}(f''Z - R) \end{bmatrix} .$$

Before examining the necessary and sufficient conditions for long-run stability, it would be helpful to have a way of ascertaining the convergence or divergence of a time path without having to solve for characteristic roots of the differential equation. Such a method exists. Samuelson has presented the Routh theorem that

gives us the necessary and sufficient conditions in order for the real parts of the roots to be negative.[1]

Routh Theorem

Consider a polynomial

$$f(\lambda) = a_o\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 ,$$

where $a_o > 0$. Then, the Routh theorem states that the necessary and sufficient conditions for the real parts of the roots to be negative is that the following sequence of determinants all be postive:

$$\Delta_1 = | a_1 | , \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_o & a_2 \end{vmatrix} , \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ a_o & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} .$$

To see the relation between a_o, a_1, a_2, a_3 and our matrix of a dynamic system, let E be a square matrix (3×3) that is non-singular:

$$E = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

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[1]P. A. Samuelson, Foundations of Economic Analysis (Cambridge: Harvard University Press, 1947), pp. 430-36.

Let us consider a matrix $[E - \lambda I]$, where λ is a scalar variable and I is the unit matrix. Then

$$[E - \lambda I] = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix}$$

$|E - \lambda I|$ is called the characteristic determinant of E. By expanding the determinant we get the following scalar polynomial:

$$\begin{aligned} f(\lambda) = |E - \lambda I| = -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 + \\ [-a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33} + a_{13}a_{31} + a_{32}a_{23} \\ + a_{12}a_{21}]\lambda + (\text{determinant } E) . \end{aligned}$$

The polynomial $f(\lambda)$ is called the characteristic function of the matrix E. Now, setting $f(\lambda) = 0$, and multiplying both sides by -1, we have

$$a_o\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 ,$$

which is the characteristic equation of matrix E, where

$a_o = 1 > 0$, $a_1 = -(a_{11} + a_{22} + a_{33}) = -$ trace of our matrix of the dynamic system,

$$a_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} ,$$

and

$$a_3 = -|E|\ , \quad \text{the determinant of the matrix of the dynamic system.}$$

Necessary and Sufficient Conditions for Long-run Stability

As we have seen, the necessary and sufficient conditions for long-run stability require that Δ_1, Δ_2, and $\Delta_3 > 0$, where

$$\Delta_1 = a_1 = -(a_{11} + a_{22} + a_{33}) = -\text{ trace} > 0 \qquad \text{i.e., } \quad \text{trace} < 0$$

$$\Delta_2 = (a_1a_2 - a_oa_3) = a_1a_2 - a_3 > 0 \qquad (\text{since } \ a_o = 1)$$

$$\Delta_3 = a_3(a_1a_2 - a_3) > 0.$$

Since $a_1a_2 - a_3$ must be greater than zero, this implies that $a_3 > 0$ must hold, i.e., a negative value of determinant is greater than zero or a determinantal value is negative. Therefore, in order to have stability, the following inequalities must hold:

(1) trace < 0,

(2) $a_2 > 0$,

(3) a determinantal value is negative, and

(4) $a_1a_2 - a_3 > 0$.

We now calculate these values.

(1) Trace $= a_{11} + a_{22} + a_{33}$

From (4.20) we have

$$a_{11} = \frac{1}{A}[D - \frac{b}{r}(1 - s)X]\ .$$

If $D < 0$, then $a_{11} < 0$, where $D = sf' - f''(1 - s)(1 + \theta_1)b - n$.

$$a_{22} = -\varepsilon b(1+\theta_1) + \frac{b}{Ar}(1+\theta_1)[f''(1-s)n - \gamma\varepsilon] < 0 ,$$

since γ and ε are greater than zero, and $f'' < 0$.

$$a_{33} = \frac{1}{A}(f''Z - R) < 0 ,$$

since $R > 0$, $Z > 0$, and $f'' < 0$. Therefore, trace $= a_{11} + a_{22} + a_{33} < 0$, i.e., $a_1 = -$ trace > 0, so long as $D < 0$.

(2) $\underline{a_2}$

Let us write a_2 as $a_2 = \alpha_1 + \alpha_2 + \alpha_3$, then

$$\alpha_1 = a_{11}a_{22} - a_{12}a_{21} = \frac{-(1+\theta_1)b}{A}\{D(1+\frac{\gamma}{r}) + (1-s)$$

$$\{\varepsilon(H_k + \hat{B}_k)(\frac{\gamma\varepsilon b}{r} - n) - \frac{\gamma}{r}[(H_k + \hat{B}_k) - f''](\varepsilon b + n)\}\}$$

$\alpha_1 > 0$ if $D < 0$, and

$$\varepsilon(H_k + \hat{B}_k)(\frac{\gamma\varepsilon b}{r} - n) - \frac{\gamma}{r}(\varepsilon b + n)[\varepsilon(H_k + \hat{B}_k) - f''] < 0$$

$$= -\varepsilon n(H_k + \hat{B}_k)(\gamma + r) + \gamma f''(\varepsilon b + n) < 0$$

$$\varepsilon n(H_k + \hat{B}_k)(\gamma + r) > \gamma f''(\varepsilon b + n) .$$

Since $f'' < 0$, $H_k > 0$, and $\hat{B}_k > 0$, this condition holds.

$$\alpha_2 = a_{11}a_{33} - a_{13}a_{31} = \frac{1}{A}\{\gamma b(1+\theta_1)(1-s)(1+\frac{n}{r})[\varepsilon(H_k + \hat{B}_k) - f'']$$

$$- D[\gamma\{\varepsilon[\frac{b}{r}(1+\theta_1) + H_r + \hat{B}_r]\} + 1]\} .$$

If $D < 0$, then $\alpha_2 > 0$.

$$\alpha_3 = a_{22}a_{33} - a_{23}a_{32}$$

$$= \frac{-\varepsilon b(1 + \theta_1)}{A} \{f''(1 - s)[b(1 + \theta_1) - n(H_r + \hat{B}_r)] - \gamma\} .$$

If $b(1 + \theta_1) - n(H_r + \hat{B}_r) \geqq 0$, then $\alpha_3 > 0$. Therefore, $a_2 = \alpha_1 + \alpha_2 + \alpha_3 > 0$ if

$$D = sf' + f''(1 - s)(1 + \theta_1)b - n < 0 ,$$

and

$$b(1 + \theta_1) - n(H_r + \hat{B}_r) \geqq 0 \quad .$$

(3) Determinant

Multiplying the third row of (4.20) by $\frac{b}{r}$ and adding it to the second row we get

$$\text{(4.21)} \qquad \text{det.} = \varepsilon b \begin{vmatrix} a_{11} & a_{12} & a_{31} \\ H_k + \hat{B}_k & -(1 + \theta_1) & \frac{b}{r}(1 + \theta_1) + H_r + \hat{B}_r \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

By expansion we get $\text{det.} = \varepsilon b(\alpha_1' + \alpha_2' + \alpha_3')$, where

$$\alpha_1' = \frac{-(1 + \theta_1)}{A} a_{11}\{f''(1 - s)[b(1 + \theta_1) - n(H_r + \hat{B}_r)] - \gamma\}$$

$$\alpha_2' = \frac{\gamma\varepsilon(1 + \theta_1)(1 - s)}{A} (H_k + \hat{B}_k)\{b(1 + \theta_1) - n(H_r + \hat{B}_r)\}$$

$$\alpha_3' = \frac{(1 + \theta_1)(1 - s)}{A} a_{31}\{b(1 + \theta_1) - n(H_r + \hat{B}_r) - \frac{b}{r}\gamma\}$$

Therefore,

$$\text{det.} = \varepsilon b(\alpha_1' + \alpha_2' + \alpha_3')$$

$$= \frac{\varepsilon b(1+\theta_1)}{A}\{f''(1-s)[b(1+\theta_1) - n(H_r + \hat{B}_r)] + \gamma D\} .$$

Therefore, det. < 0, if $D < 0$, and

$$b(1+\theta_1) - n(H_r + \hat{B}_r) \geqq 0 \quad ,$$

i.e., $a_3 = -$ det. > 0.

(4) $\underline{a_1a_2 - a_3 = a_1a_2 + \text{det.}}$

$$a_1a_2 - a_3 = -(a_{22}+a_{33})\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix}\right.$$

$$\left. + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix}\right\} - a_{11}\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix}\right\}$$

$$- a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33}\end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix} .$$

Since $a_2 > 0$, a_{11}, a_{22}, and $a_{33} < 0$, we have the sum of the first five terms greater than zero. Therefore, $a_1a_2 - a_3 > 0$, if

$$- a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33}\end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix} > 0 .$$

We note that this expression is $\varepsilon b(\alpha_2' + \alpha_3')$, i.e., the last two parts of the determinant. Therefore, we can write

$$\varepsilon b(\alpha_2' + \alpha_3') = \frac{1}{A^2}\,\varepsilon b(1 + \theta_1)(1 - s)\{\gamma\varepsilon A(H_k + \hat{B}_k)[b(1 + \theta_1) - n(H_r + \hat{B}_r)]$$

$$+ [\frac{\gamma b}{r} - \{b(1 + \theta_1) - n(H_r + \hat{B}_r)\}]$$

$$\times [\gamma\{\varepsilon(H_k + \hat{B}_k) - f''\} - f''D]\} \quad .$$

$a_1a_2 - a_3$ will be positive if the following conditions hold:

(1) if (a) $\frac{\gamma b}{r} - [b(1 + \theta_1) - n(H_r + \hat{B}_r)] < 0$, and

(b) $\gamma[\varepsilon(H_k + \hat{B}_k) - f''] - f''D < 0$,

or

(2) if (a) $\frac{\gamma b}{r} - [b(1 + \theta_1) - n(H_r + \hat{B}_r)] > 0$, and

(b) $\gamma[\varepsilon(H_k + \hat{B}_k) - f''] - f''D > 0$.

Condition (1) does not hold when γ and $\varepsilon \to \infty$, but condition (2) holds. Collecting these results thus far, the necessary and sufficient conditions for long-run stability are:

$$a_1 = -\text{ trace} > 0 \qquad \text{if} \qquad D < 0 \ ,$$

$$a_2 > 0 \qquad \text{if} \qquad b(1 + \theta_1) - n(H_r + \hat{B}_r) \geqq 0 \ ,$$

$$a_3 = -\text{ det.} > 0 \qquad \text{if} \qquad D < 0$$

$$b(1 + \theta_1) - n(H_r + \hat{B}_r) \geqq 0 \ ,$$

$$a_1a_2 - a_3 > 0 \qquad \text{if} \qquad \frac{\gamma b}{r} > b(1 + \theta_1) - n(H_r + \hat{B}_r)$$

$$\gamma[\varepsilon(H_k + \hat{B}_k) - f''] > f''D \ .$$

The required conditions for long-run stability, then, are summarized in the following three conditions:

$$(4.22) \qquad sf' - n < 0$$

$$(4.23) \qquad \frac{\gamma b}{r} > b(1 + \theta_1) - n(H_r + \hat{B}_r) \geqq 0$$

$$(4.24) \qquad \gamma[\varepsilon(H_k + \hat{B}_k) - f''] > f''D \ .$$

Or equivalently, we may rewrite the above conditions as follows:

$$(4.22)' \qquad n - sf' > 0 \ ,$$

$$(4.23)' \qquad \gamma > r(1 + \theta_1) - n(\eta_b + \eta_m \theta_1) \geqq 0 \ ,$$

$$(4.24)' \qquad \gamma[\varepsilon B_k(1 + \theta_1) - f''] > f''D \ ,$$

where $H_r + \hat{B}_r = (\eta_b + \eta_m \theta_1) \frac{b}{r}$, and $\theta_1 \hat{B}_k = H_k$. If $\gamma, \varepsilon \rightarrow \infty$ the long-run equilibrium is stable, so long as $n - sf' > 0$. In other words, if people adjust their expectations very rapidly (i.e., the "expectation coefficient" is very large), and if the speed of adjustment in asset markets is very high, the long-run equilibrium is stable so long as the natural rate of growth is greater than the product of the propensity to save and the rate of return on capital --- that is, the difference of the slopes of the natural growth rate and the savings function is greater than zero. If $\gamma \rightarrow \infty$, the long-run equilibrium is stable, so long as

$n - sf' > 0$. If $(\eta_b + \eta_m \theta_1) = 0$, i.e., $\theta_1 = -\frac{\eta_b}{\eta_m}$, and if either γ or ε, or both, approaches infinity, the long-run equilibrium is stable, provided $n - sf' > 0$. From condition (4.23)' it is obvious that there exists a range for the government debt ratio, or equivalently,

$$r(b + m) \geqq n(b\eta_b + m\eta_m) \; , \tag{4.25}$$

within which the long-run equilibrium is stable; the limits of the range depend on the natural rate of growth, the market rate of interest, and the demand elasticities for money and bonds with respect to the rate of interest. Hadjimichalakis has shown that the generalized Tobin model is locally unstable in the short-run and long-run when $\gamma, \varepsilon \to \infty$. But we have found that when we introduce government bonds into the model, the short-run and long-run dynamic systems are stable under the condition(s) mentioned above.

CHAPTER V

CONCLUSIONS

In this thesis an attempt has been made to introduce a third asset into the neoclassical monetary growth model. Government bonds, an alternative asset to real capital and money, was introduced in a very simple way. The effect of government bonds upon the real variables of the economy was analyzed through the change in the rate of issue of bonds by the monetary authority. We have found that the alternative asset is not neutral, in the sense that a change in the rate of its growth affects the real sector of the economy. The alternative asset is interest-bearing, and therefore the interest rate is given scope within which to play its role. That part of disposable income which is generated by interest payment has been included, and disposable income has crucial role in our three asset one-sector growth model, as it does in most of the neoclassical monetary growth models.

The introduction of government bonds into the monetary growth model, as an alternative asset to real capital and money, raised certain questions as to whether or not there exist equilibrium values for each variable, and whether or not those values are stable.

Throughout this study, we have held three crucial assumptions. The first is that the anticipated rate of interest, $\rho + \pi$, be equal to the observed market rate on government bonds, r,

which consists of two components, namely, the real rate of return on physical capital and the expected rate of change in the price level. This assumption can be relaxed, and the demand functions for money and bonds can then be written as functions of three variables rather than two, namely, the capital-labour ratio, r^o (if we treat r^o as the real market rate of interest), and π. The analysis can be carried out, but, since this would entail a 4×4 matrix, we have avoided it in order not to make the model more complicated. The second assumption is that government expenditures on goods and services are equal to the government revenue from taxation, so that government transfer and interest payments are entirely financed by the creation of new issues of money and bonds. The relaxation of this assumption would require a government budget constraint which would make the model more complicated than the one we have been using.[1] To include the government budget constraint would require a more general model which would have to consider the monetary effects of fiscal policy or the fiscal effects of monetary policy. The third assumption is that, the rate of money expansion is equal to the rate of bond expansion. The removal of this assumption has been attempted. The rate of growth of the financial

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[1] C. F. Christ, "A Short-run Aggregate-Demand Model of the Interdependence and Effects of Monetary and Fiscal Policies with Keynesian and Classical Elasticities," American Economic Review 57 (May 1967): 434-73. See also C. F. Christ, "A Simple Macroeconomic Model with Government Budget Restraint," Journal of Political Economy 76 (January/February 1968): 53-68.

market could be written as $g = \frac{\mu m + \delta b}{m + b}$.[1] We have assumed that $g = \theta$, i.e., $\mu = \delta$, where μ and δ denote the rate of growth of money and bonds respectively. This indicates that the government debt ratio is not constant, but we are not interested in the effect of the long-run values of the rate of growth of the financial market and the government debt ratio on the nature of the long-run equilibrium. Our purpose, as we have mentioned earlier, is rather to examine the stability properties, if any, of both the short-run and long-run dynamic system.

Whether or not this study has been successful depends upon whether or not one accepts the specification of the model and the assumptions as reasonable. If they are considered reasonable, we have achieved the following results. Defining short-run equilibrium as a temporary equilibrium for a time period in which the capital and labour force are given, we have found a set of necessary and sufficient conditions for local stability of the short-run dynamic model. The set of conditions depends on the elasticities of demand for money and bonds with respect to the rate of interest. The short-run equilibrium is stable if the desired ratio of the government debt, money to bonds, is less than, or equal to, the negative of the ratio of the demand elasticities for bonds to money (in terms of the rate of interest). If the monetary authorities are aware of this ratio, they can choose the government debt ratio in such a way that the

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[1] W. Ethier, "Financial Assets and Economic Growth in a Keynesian Economy," Journal of Money, Credit, and Banking 7 (May 1975): 219.

economy reaches a stable short-run equilibrium. For example, if the interest elasticity of demand for bonds is equal to the absolute value of the interest elasticity of demand for money, we have $\theta_1 \leqq 1$, i.e., $m \leqq b$. If the above condition does not hold for the short-run dynamic model, or, in other words, if the desired ratio of money to bonds is greater than the negative of the ratio of the demand elasticities for money and bonds, the short-run equilibrium is stable provided that $\theta_1 < -\frac{\eta_b + 1}{\eta_m + 1}$. However, there exists a range for the government debt ratio within which the short-run equilibrium is stable; the limits of the range depend on the magnitude of these demand elasticities. The above conditions hold even though the speed of adjustment in asset markets is high and people adjust their expectations rapidly. Since the short-run dynamic model does not take into account the effects of changes in the market rate of interest on the real sector of the economy, we have next examined the long-run stability conditions.

We have found that the long-run equilibrium value of money per capita, m^*, in our three asset one-sector economy is lower than that in the neoclassical monetary growth model without government bonds. We also saw that the capital-labour ratio in this model is lower than that in one-sector growth models without money and bonds. The reason is that our model contains the alternative assets in the portfolio of the private sector. On the question of long-run stability, we have found a set of necessary and sufficient conditions for the long-run dynamic model. We saw

that if expectations and asset markets adjust very rapidly, the long-run equilibrium is stable so long as the difference of the slopes of the natural growth rate and savings function is greater than zero. Our results refute those of the Hadjimichalaskis' "Generalized Tobin Model" which claims that the short-run and long-run dynamic system is locally unstable when the expectation coefficient and the speed of adjustment in asset markets are high. One of the main reasons would be that ignoring the rate of interest in an analysis forces the price level alone to carry the burden of stability proper through behavioural parameters in asset markets. The long-run analysis in this paper, however, has demonstrated that there exists a range for the government debt ratio within which the long-run equilibrium is stable, and the limits of the range depend on the natural rate of growth, the market rate of interest, and the interest elasticities of demand for money and bonds:

$$\theta_1 \geqq \frac{n\eta_b - r}{r - n\eta_m} .$$

This condition is consistent with the results following from open market operations, since an open market sale of bonds will raise the market rate of interest and lower the ratio of money to bonds, θ_1. Finally, our results also show that, when we introduce government bonds into the neoclassical monetary growth model, and when $\gamma \rightarrow \infty$, the conclusion reached by Sidrauski[1] --- i.e., the higher the

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[1]M. Sidrauski, "Inflation and Economic Growth," Journal of Political Economy 75 (December 1967): 807.

expectation coefficient the higher the probability of the system being unstable -- no longer holds. Our conclusion, contrary to the established fact, may critically depend on the specification we have imposed on the actual rate of change in the price level; i.e., Eq (3.7), p. 25.

There is much work yet to be done in the area of monetary growth models. For example, it would be useful to analyze a model which includes a government budget constraint, a savings function with wealth as well as disposable income as arguments, an open economy rather than a closed one, and inside money as well as outside money.

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